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Entropy Dissipation and Moment Production for the Boltzmann Equation

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Let $H(f \mid M) = \int f \log(f/M)\, dv$ be the relative entropy of f and the Maxwellian with the same mass, momentum, and energy, and denote the corresponding entropy dissipation term in the Boltzmann equation by $D(f) = \int Q(f, f) \log f\, dv$. An example is presented which shows that $|D(f)/H(f \mid M)|$ can be arbitrarily small. This example is a sequence of isotropic functions, and the estimates are very explicitly given by a simple formula for D which holds for such functions. The paper also gives a simplified proof of the so-called Povzner inequality, which is a geometric inequality for the magnitudes of the velocities before and after an elastic collision. That inequality is then used to prove that $\int f(v)\, |v|^s\, dt < C(t)$, where f is the solution of the spatially homogeneous Boltzmann equation. Here $C(t)$ is an explicitly given function depending s and the mass, energy, and entropy of the initial data.

KEY WORDS: Boltzmann equation; entropy production; Povzner inequality; moments.

1. INTRODUCTION

This paper concerns the spatially homogeneous Boltzmann equation

$$\partial_t f(v, t) = Q(f, f)(v, t) \tag{1.1}$$

where $Q(f, f)$ is the so-called collision operator,

$$Q(f, f)(v) = \int_{\mathbb{R}^3} \int_{S^2} (f' f'_1 - f f_1)\, B(\omega \cdot (v - v_1)\, |v - v_1|^{-1}, |v - v_1|)\, d\omega\, dv_1$$

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Here $f=f(v)$, $f_1=f(v_1)$, $f'=f(v')$, and $f'_1=f(v'_1)$, and v' and v'_1 are the velocities after the collision of two particles which had the velocities v and v_1, respectively, before they collided. The collisions are assumed to conserve mass, $v+v_1=v'+v'_1$, and energy, $v^2+v_1^2=v'^2+v_1'^2$. Hence v' and v'_1 are situated opposite to each other on the sphere with radius $|v-v_1|$ and center in $(v+v_1)/2$, and one way of writing the relations connecting the four involved velocities is

$$v'=\frac{v+v_1}{2}+\frac{|v-v_1|}{2}\omega$$

$$v'_1=\frac{v+v_1}{2}-\frac{|v-v_1|}{2}\omega$$

Here ω is the unit vector in the direction from $(v+v_1)/2$ to v' (cf. Fig. 1). This particular parametrization of the gain term can be found in, e.g., ref. 4. The collision operator is thus an average over all possible collisions that can take place, and B is a weight factor giving the effect of a particular collision. We consider here only the case where the particles interact by inverse-power-law potentials, in which case B factorizes into a function of the form $B=h(\theta)\,|v-v_1|^\beta$. When the particles are hard spheres, $B=|v-v_1|$, and this is generalized to the so-called variable-hard-spheres model, where $B=|v-v_1|^\beta$ and $\beta\geqslant 0$.

The theory of the Boltzmann equation is treated, e.g., in ref. 7. The equation is known to be well posed under very general conditions on the initial data. As a consequence of the conservation of mass, momentum, and energy in a single collision, the total mass, momentum, and energy of the gas are constant in time. In terms of the solutions f, this is manifested in that the first moments

$$\int_{\mathbb{R}^3} f(v,t)\,dv, \qquad \int_{\mathbb{R}^3} f(v,t)\,v\,dv, \qquad \int_{\mathbb{R}^3} f(v,t)\,|v|^2\,dv \tag{1.2}$$

are conserved. The stationary solutions are the Maxwellians, i.e., functions of the form $a\exp(-|v-v_0|^2/b)$, and there is a unique Maxwellian corresponding to the conserved quantities (1.2); it is known that the solutions f converge strongly to this Maxwellian. This is related to the fact that the entropy

$$\int_{\mathbb{R}^3} f(v)\log(f(v))\,dv \tag{1.3}$$

is monotonously decreasing.

It is convenient to rescale the problem so that the mass is one, the momentum vanishes, and the energy is equal to three. The unique Maxwellian associated with these moments is $M(v) = (2\pi)^{-3/2} \exp(-|v|^2/2)$. The relative entropy with respect to M is defined as

$$H(f \mid M) = \int f(v) \log \frac{f(v)}{M(v)} \, dv$$

and the entropy dissipation term is

$$D(f) \equiv \frac{d}{dt} H(f \mid M) = \int Q(f, f)(v) \log f(v) \, dv \tag{1.4}$$

The entropy dissipation term is negative, and vanishes if and only if f is a Maxwellian, and the relative entropy vanishes if and only if $f = M$. It was suggested in ref. 6 that an inequality of the type $|D(f)| \geqslant C\,|H(f, M)|$ could hold, with C depending only on the collision kernel B and on the Maxwellian M. An estimate of that type would imply that the solutions of the Boltzmann equation converge exponentially to equilibrium at a rate depending only on the mass and energy of the initial data. The inequality would also be useful in the study of various limit problems for the full Boltzmann equation.

An example showing that the inequality suggested above cannot hold was provided by Bobylev[3] (see also ref. 5) for the case of Maxwellian molecules ($\beta = 0$). He constructed initial data for (1.1) such that the solutions tend to equilibrium exponentially, but at an arbitrarily slow rate. Here we study the relative entropy and the entropy dissipation directly and give an example showing that also in the case of hard potentials ($\beta > 0$) the inequality cannot be as general as conjectured. Such examples can be provided by abstract arguments (this will be discussed in Section 3), but it is also possible to make rather precise estimates for isotropic functions (i.e., depending only on $|v|$). A simple formula for the entropy dissipation for isotropic functions is derived in Section 2, and in Section 3 this formula is used to obtain the desired estimates.

The geometric arguments used in Section 2 are also used in Section 4 to give a simple proof of a slightly generalized form of the Povzner inequality (see ref. 9 and the references therein). The Povzner inequality gives estimates on the rate of change of radial moments (i.e., of $\int f(v)\,|v|^s\,dv$) of solutions of the Boltzmann equation. In particular, they can be used to prove that, at least for hard potentials, all moments that are initially bounded remain bounded globally in time. Such results, which can be found in, e.g., refs. 1 and 9, are important in the analysis of solutions of the Boltzmann equation, since uniform bounds on moments with $s>2$ can be used for proving that the energy of the solutions is conserved. The fact that there is still no proof of energy conservation for solutions of the space-dependent equation depends on the lack of estimates on the evolution of higher moments in that case.

An important improvement of Elmroth's result was obtained by Desvillettes,[8] who proved that if the initial data possess moments of order strictly larger than 2, then $\int f(v,t)\,|v|^s\,dv$ is bounded for all $t>0$ and $s\geqslant 0$. This result holds for $\beta>0$; in the Maxwellian case, the statement is false (a proof that this is the case can be found in ref. 10, which also contains a simplification of the proof from ref. 8 and analogous L^p estimates). The last result of the present paper, Theorem 4.2, states that (still for $\beta>0$) it suffices to demand that the quantities (1.2), (1.3) are bounded initially. The proof of Theorem 4.2 is a direct calculation, which also gives good estimates on the constants involved.

2. THE ENTROPY DISSIPATION FOR ISOTROPIC DISTRIBUTIONS

The entropy dissipation can be written

$$-\frac{1}{4}\iiint (f'f'_1-ff_1)\log\frac{f'f'_1}{ff_1}\,h(\theta)\,|v-v_1|^\beta\,d\omega\,dv\,dv_1 \tag{2.1}$$

To derive this expression from (1.4), one can, at least formally, use the change of variables $dv\,dv_1\to dv'\,dv'_1$. Since the gas is assumed to be isotropic, f, f_1, f', and f'_1 depend only on the magnitudes of the vectors v, v_1, v', and v'_1, which will be denoted r, r_1, r', and r'_1, respectively. The notation is described in Fig. 1. To find expressions for r' and r'_1 it is useful to introduce polar coordinates in (r, r_1) by defining

$$r=\rho\cos\gamma,\qquad r_1=\rho\sin\gamma$$

Moreover, let $p=(v+v_1)/2$. Then, by the cosine theorem

$$|v-v_1|=\sqrt{r^2+r_1^2-2rr_1\cos\theta_1}=\frac{\rho}{2}\sqrt{1-\sin 2\gamma\cos\theta_1}$$

and

$$|p|=\frac{1}{2}\sqrt{r^2+r_1^2+2rr_1\cos\theta_1}=\frac{\rho}{2}\sqrt{1+\sin 2\gamma\cos\theta_1}$$

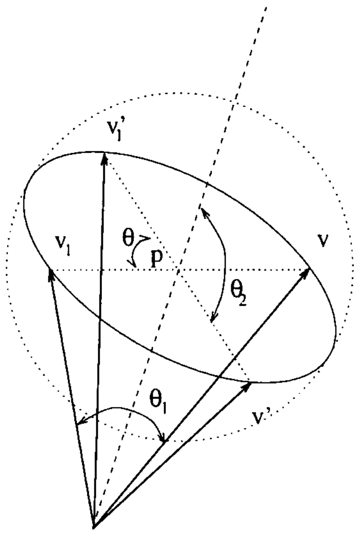

Fig. 1. Geometry of a binary collision.

Then a second application of the cosine theorem and some simplification gives

$$
\begin{aligned}
r' &= \frac{\rho}{\sqrt{2}} [1 + \cos\theta_2 (1 - \sin^2 2\gamma \cos^2\theta_1)^{1/2}]^{1/2} \\
r'_1 &= \frac{\rho}{\sqrt{2}} [1 - \cos\theta_2 (1 - \sin^2 2\gamma \cos^2\theta_1)^{1/2}]^{1/2}
\end{aligned}
\tag{2.2}
$$

In the change of variables in (2.1) we can now fix a polar axis in the direction of v, and the rotational symmetry in the problem then gives $dv \to 4\pi r^2\, dr$, $dv_1 \to 2\pi \sin\theta_1\, d\theta_1\, r_1^2\, dr_1$, and $d\omega \to \sin\theta_2\, d\theta_2\, d\phi$. The result is

$$
\begin{aligned}
&-\pi \int_0^\infty r^2\, dr \int_0^\infty r_1^2\, dr_1\, 2\pi \int_0^\pi \sin\theta_1\, d\theta_1 \\
&\quad \times \int_0^\pi \sin\theta_2\, d\theta_2 \int_0^{2\pi} h(\theta)\, d\phi\, \frac{\rho^\beta}{2^\beta} (1 - \sin 2\gamma \cos\theta_1)^{\beta/2}\, \Psi(r, r_1, r', r'_1)
\end{aligned}
\tag{2.3}
$$

where Ψ denotes the expression involving f. In general θ is a complicated (but explicit) expression of the remaining variables, but in the case of (variable) hard spheres, where $h(\theta) = 1$, the ϕ integral can be computed;

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the result is 2π. For the sake of simplicity, only that case will considered henceforth. The final expression is also simplified by the fact that Ψ is symmetric with respect to the first two variables as well as with respect to interchange of the primed and unprimed variables.

Changing to polar coordinates in the (r, r_1) integral and then writing

$$t = \cos\theta_1, \qquad u = \cos\theta_2$$

gives [r' and r'_1 are given by (2.2)]

$$-\frac{\pi^2}{2^{1+\beta}}\int_0^\infty \rho^{5+\beta}\,d\rho \int_0^{\pi/2} \sin^2 2\gamma\,d\gamma$$

$$\times\int_{-1}^{1} dt \int_{-1}^{1} du \int_0^{2\pi} h(\theta)\,d\phi\,(1 - t\sin 2\gamma)^{\beta/2}\,\Psi(\rho\cos\gamma, \rho\sin\gamma, r', r'_1)$$

A new change of variables,

$$\cos 2\gamma \to s, \qquad t\sqrt{1-s^2} \to t, \qquad u\sqrt{1-t^2} \to u$$

together with the symmetries of Ψ, gives

$$-\frac{\pi^3}{2^\beta}\int_0^\infty \rho^{5+\beta}\,d\rho \int_0^1 ds \int_0^{\sqrt{1-s^2}} dt \int_0^{\sqrt{1-t^2}} du\, \frac{(1-t)^{\beta/2} + (1+t)^{\beta/2}}{\sqrt{1-t^2}}$$

$$\times\,\Psi\left(\rho\sqrt{\frac{1+s}{2}}, \rho\sqrt{\frac{1-s}{2}}, \rho\sqrt{\frac{1+u}{2}}, \rho\sqrt{\frac{1-u}{2}}\right) \tag{2.4}$$

After a change of the order of integration in t and u, the inner part of the integral becomes

$$\int_0^s \int_0^{\sqrt{1-s^2}} \frac{(1-t)^{\beta/2} + (1+t)^{\beta/2}}{\sqrt{1-t^2}}\,dt\,\Psi(r, r_1, r', r'_1)\,du$$

$$+\int_s^1 \int_0^{\sqrt{1-u^2}} \frac{(1-t)^{\beta/2} + (1+t)^{\beta/2}}{\sqrt{1-t^2}}\,dt\,\Psi(r, r_1, r', r'_1)\,du$$

The t integrals can be evaluated, but the result can be expressed in terms of elementary functions only if $\beta = 0$ or $\beta = 1$. For $\beta = 1$, the result is

$$2^{3/2}\sqrt{1-s} \qquad \text{and} \qquad 2^{3/2}\sqrt{1-u}$$

in the first and second terms, respectively. For all $\beta \in [0, 1]$, the asymptotic behavior near $s = 1$ (or $u = 1$) is similar, and therefore this expression will be used also when the β dependence is kept in the power of ρ. We thus arrive at

$$\int_0^1 \left(\int_0^s \sqrt{1-s}\, \Psi(r, r_1, r', r'_1)\, du + \int_s^1 \sqrt{1-u}\, \Psi(r, r_1, r', r'_1)\, du \right) ds$$

Finally we note that because Ψ is symmetric with respect to the primed and unprimed variables, the two terms are equal, and hence

$$D(f) = -\pi^3 2^{5/2-\beta} \int_0^\infty \rho^{5+\beta} \int_0^1 \int_0^s \sqrt{1-s}\, \Psi(r, r_1, r', r'_1)\, du\, ds\, d\rho \quad (2.5)$$

where the arguments of Ψ are as in (2.4).

3. ARBITRARILY SMALL ENTROPY DISSIPATION

In this section, the formula from Section 2 is used to construct a sequence of functions such that the ratio of the relative entropy and the entropy dissipation decreases to zero. The calculations are easy because we only consider isotropic distributions, but it is important to understand that the mechanism behind this result is independent of whether the distributions are isotropic or not. Instead the idea is the following.

Since the entropy dissipation vanishes for *any* Maxwellian, but the entropy relative to a given Maxwellian is zero only for exactly the same Maxwellian, it is natural to attempt a counterexample in the form of a sequence of functions f_n with given first moments (so that the Maxwellian M_a with which to compute the relative entropy is fixed) which converges pointwise to a Maxwellian M_b with different moments. In fact, the example by Bobylev[3] is of that form. Since the integrand of the relative entropy is positive, it follows immediately that $\lim_{n \to \infty} H(f_n \mid M_a) \geq H(M_b \mid M_a) > 0$. This integral involves a factor $|v|^2$ which comes from $\log M_a$, and therefore this means exactly that $\int f_n(v)\, |v|^s\, dv$ is uniformly convergent for all $s < 2$, but not for $s = 2$. Now, the integrands in the entropy dissipation term converge pointwise to 0 as $n \to \infty$, and therefore one only needs to establish that the integrals converge uniformly. In practice that means imposing bounds from below on the sequence f_n. Note, for example, that if $f_n(v) = 0$ for all v in some open set, then the entropy dissipation rate is infinite, regardless of the values of f_n for other v.

For an example, let $M_T(r) = (2\pi T)^{-3/2} \exp(-r^2/2T)$, and define

$$f_\varepsilon(v) = M_1(|v|) + \varepsilon(1+|v|)^{-(5+\varepsilon)} \equiv M_1(|v|) + g_\varepsilon(|v|) \tag{3.1}$$

Then, as $\varepsilon \to 0$, $\int_{\mathbb{R}^3} f_\varepsilon(v)\, dv \to 1$ and $\int_{\mathbb{R}^3} f_\varepsilon(v)\, |v|^2\, dv \to 3+1$. The Maxwellian with the corresponding mass and energy is $M_{4/3}(v)$. Then $|D(f_\varepsilon)/H(f_\varepsilon \mid M_{4/3})| \to 0$ when $\varepsilon \to 0$, because

$$H(f_\varepsilon \mid M_{4/3}) \to \int_{\mathbb{R}^3} M_1(|v|) \log(M_1(|v|)/M_{4/3}(|v|)\, dv > 0$$

and

$$|D(f_\varepsilon)| \to 0$$

The first expression follows from the fact that the family f_ε is uniformly bounded from below by M_1, uniformly bounded from above by, e.g., Cg_0, and pointwise convergent. Since the integrand of $D(f_\varepsilon)$ is pointwise converging to zero, all that is needed to prove the second one is to establish some uniform integrability. To do that, insert (3.1) into

$$\Psi(r, r_1, r', r'_1) = (f'f'_1 - ff_1)[\log(f'f'_1) - \log(ff_1)]$$

Since $M(r)\, M(r_1) = M(r')\, M(r'_1)$, only terms of order ε remain after expanding the expression for Ψ. All these terms can be handled in a similar way, and here we only study one of them:

$$\begin{aligned} &|g_\varepsilon(r)\, g_\varepsilon(r_1) \log[f_\varepsilon(r')\, f_\varepsilon(r'_1)]| \\ &\quad \leqslant g_\varepsilon(r)\, g_\varepsilon(r_1)\, |\log[g_\varepsilon(r)\, g_\varepsilon(r_1)]| \\ &\quad \leqslant \varepsilon(1+r)^{-(5+\varepsilon)} (1+r_1)^{-(5+\varepsilon)} \log[\varepsilon^2(1+r')^{-(5+\varepsilon)} (1+r'_1)^{-(5+\varepsilon)}] \\ &\quad \leqslant C\, |\varepsilon \log \varepsilon|\, (1+\rho)^{-5+\delta} (1+\sqrt{1-u\rho})^{-5} \end{aligned}$$

The term has been simplified here by taking into account that $M(r) \leqslant c(1+r)^{-5}$, and also that outside the log term an upper bound is desired and that a lower bound is needed for the argument of the logarithm. Then the log term is estimated by a δ power. Next this expression is inserted into (2.5) and integrated over $\rho^{5+\beta}\, d\rho$. The remaining integrand is of the form

$$C(1-u)^{-(1+\beta+\delta)/2}$$

and carrying out the last integration, one sees that this part of the entropy dissipation is $\mathcal{O}(\varepsilon \log \varepsilon)$. In the same way it follows that all terms converge to zero as ε decreases.

This construction fails if the sequence f_ε has some higher order moment uniformly bounded. The result in Section 4 shows that counterexamples of the type given here are not very relevant for solutions to the space-independent Boltzmann equation, at least not for hard potentials. Thus it would be interesting to know whether the inequality proposed by Cercignani holds for a smaller class of functions. This question has been partially answered by Carlen and Carvalho (e.g., ref. 5), who proved that for all f with $\int f(v)\,|v|^s\,dv < C < \infty$, $s > 2$, there is a strictly increasing function ϕ such that

$$-D(f) \geqslant \phi(H(f \mid M))$$

Though this has not been proven, it is conceivable that ϕ grows linearly near the origin.

4. ABOUT POVZNER'S INEQUALITY AND MOMENT GENERATION

The expressions for $|v|$, $|v_1|$, $|v'|$, and $|v'_1|$ given in Section 2 can be used to obtain a simple proof of Povzner's inequality, which is an essential inequality for studying the behavior of the moments of solutions of the Boltzmann equation. An improvement of that inequality was proven and used by Elmroth[9] to establish that all moments that exist initially remain bounded, and by similar methods Desvillettes[8] proved that if any moment of order higher than two exists initially, then all moments exist for any positive time.

This result holds only for molecules harder than Maxwellian, but the short proof below shows that one does not need to assume more than bounded enery and entropy for the result to be valid. That in turn implies that the example given in Section 3 is not relevant for any solution of the spatially homogeneous Boltzmann equation, except possibly for a fixed initial time interval.

Here is first a sharpened version of the Povzner inequality and the simple proof.

Theorem 4.1. Let $\psi(x) = \int_0^x \phi(\xi)\,d\xi$, where ϕ is a strictly increasing and positive function. Then

$$\psi(|v'|^2) + \psi(|v'_1|^2) - \psi(|v|^2) - \psi(|v_1|^2) \leqslant |v|^2\,\phi(|v|^2 + |v_1|^2) \qquad (4.1)$$

For $\psi(x) = |x|^{s/2}$, this becomes

$$|v'|^s + |v'_1|^s - |v|^s - |v_1|^s \leqslant C_s\,|v|^{s-1}\,|v_1| \qquad (4.2)$$

and, moreover, if $|(v'-v_1')(v-v_1)|\leqslant(1-6\varepsilon)\,|v-v_1|^2$, then

$$|v'|^s+|v_1'|^s-|v|^s-|v_1|^s\leqslant 2C_s\,|v|^{s-1}\,|v_1|-K_{s,\varepsilon}(|v|^s+|v_1|^s) \tag{4.3}$$

where $C_s=s2^{s/2-2}$ and $K_{s,\varepsilon}=s\varepsilon/4-\mathcal{O}(\varepsilon^2)-\mathcal{O}(\varepsilon^{s/2})$.

Remark. The proof below directly shows that the first two terms of (4.1) can be uniformly estimated by the remaining terms. The importance of the inequality comes from the fact that this sum then can be estimated by a product as in the theorem, and there is a considerable freedom in trading between growth in $|v|$ and in $|v_1|$. Elmroth's version of the Povzner inequality consists of the two second inequalities.

Proof. In the expressions for $|v|$, $|v_1|$, $|v'|$, and $|v_1'|$, write $\cos(2\gamma)=t$, $\cos^2\theta_1=a^2$, and $\cos\theta_2=b$. Then

$$\psi(|v|^2)+\psi(|v_1|^2)\equiv g(t)=\psi(\rho^2(1+t)/2)+\psi(\rho^2(1-t)/2)$$
$$\psi(|v'|^2)+\psi(|v_1'|^2)=g(b\sqrt{1-(1-t^2)\,a^2})$$

Both functions are even and increasing with t, and since $|b|\leqslant 1$ and $|a|\leqslant 1$, we have

$$\begin{aligned}
&\psi(|v'|^2)+\psi(|v_1'|^2)-\psi(|v_1|^2)-\psi(|v_1|^2)\\
&\leqslant g(1)-g(t)=\int_t^1 g'(\tau)\,d\tau\\
&\leqslant\frac{\rho^2}{2}\int_t^1\left[\phi\left(\frac{\rho^2(1+\tau)}{2}\right)-\phi\left(\frac{\rho^2(1-\tau)}{2}\right)\right]d\tau\\
&\leqslant\frac{\rho^2}{2}(1-t)\,\phi(\rho^2)
\end{aligned} \tag{4.4}$$

and expressed in terms of $|v|$ and $|v_1|$, this is exactly the estimate (4.1). In the special case of $\psi(x)=x^{s/2}$, this gives

$$\begin{aligned}
&|v'|^s+|v_1'|^s-|v|^s-|v_1|^s\\
&\leqslant 2^{s/2-1}\frac{s}{2}\left(\frac{\rho}{\sqrt{2}}\right)^s(1-t)\\
&\leqslant 2^{s/2}\frac{s}{4}\left(\frac{\rho}{\sqrt{2}}\right)^s(1-t)^{1/2}\,(1+t)^{(s-1)/2}=2^{s/2}\frac{s}{4}\,|v|\cdot|v_1|^{s-1}
\end{aligned}$$

which is the second inequality in Theorem 4.1. Next we note that if $t < 1 - \varepsilon$ and $\varepsilon < 1/2$, then

$$(1-t) = (1-t)\, g(t)/g(t) > \varepsilon g(t)/g(1-\varepsilon)$$

and so

$$\begin{aligned} &|v'|^s + |v_1'|^s - |v|^s - |v_1|^s \\ &\quad \leqslant 2^{s/2} \frac{s}{4} \left(\frac{\rho}{\sqrt{2}}\right)^s \left[2(1-t) - \frac{\varepsilon g(t)}{g(1-\varepsilon)}\right] \\ &\quad \leqslant 2^{s/2} \frac{s}{2} |v| \cdot |v_1|^{s-1} - \varepsilon \frac{s}{4} (|v|^s + |v_1|^s) \end{aligned} \tag{4.5}$$

On the other hand, if $|b| < 1 - \varepsilon$, then

$$g(|b|) - g(t) \leqslant \frac{g(1-\varepsilon)}{g(1)} [g(1) - g(t)] - \frac{g(1-\varepsilon) - g(1)}{g(1)} g(t)$$

and

$$[g(1-\varepsilon) - g(1)]/g(1) = (1-\varepsilon/2)^{s/2} + (\varepsilon/2)^{s/2} - 1 = -s\varepsilon/4 + \mathcal{O}(\varepsilon^2) + \mathcal{O}(\varepsilon^{s/2})$$

and this implies that (4.5) holds also in this case, apart from constant factors of order one. Finally, it is easy to see that if $|\cos\theta| \leqslant (1 - 6\varepsilon)$, i.e., the condition required for (4.3) (see Fig. 1), then either $t < 1 - \varepsilon$ or $|b| < 1 - \varepsilon$.

As an application of the Povzner inequality, we analyze the evolution of moments of solutions to the Boltzmann equation, as announced in the introduction.

Theorem 4.2. Let $f_0(v)$ be given, with mass 1, energy E_0, and entropy H_0, and consider the solution $f(v, t)$ of (1.1) with $\beta > 0$ and with initial data f_0. Then for any $t > 0$,

$$\int_{\mathbb{R}^3} f(v, t)\, |v|^s\, dv \leqslant \left(\frac{A}{B[1 - \exp(-At)]}\right)^{s/\beta}$$

where the constants A and B depend on s, β, E_0, and H_0.

Proof. We recall first that mass is conserved (and is assumed to be one) and that the energy and entropy of $f(v, t)$ are bounded by E_0 and H_0, the corresponding estimates of the initial data. In particular this implies that

$$\int f(v_1, t)\,|v-v_1|^\beta\,dv \geqslant c(1+|v|^\beta) \tag{4.6}$$

where c depends only on m_0, E_0, and H_0 (cf., e.g., ref. 2). The equation for moments of order s is

$$\begin{aligned}\frac{d}{dt}&\int f(v,t)\,|v|^s\,dv\\ &=\int Q(f,f)(v)\,|v|^s\,dv\\ &=\frac{1}{2}\iiint f(v,t)\,f(v_1,t)\\ &\quad\times(|v'|^s+|v_1'|^s-|v|^s-|v_1|^s)\,|v-v_1|^\beta\,h(\theta)\,d\omega\,dv_1\,dv\end{aligned}$$

This expression can be obtained by using the same change of variables as when deriving (2.1). Write $h(\theta)=h_1(\theta)+h_2(\theta)$, where $h_1(\theta)=0$ when $\cos\theta\leqslant 2\varepsilon$. The Povzner inequality then gives

$$\begin{aligned}\frac{d}{dt}&\int f(v,t)\,|v|^s\,dv\\ &\leqslant C_s\iint f(v)\,f(v_1)\,|v_1|^{s-1}\,|v|\cdot|v-v_1|^\beta\,dv\,dv_1\\ &\quad -K_{s,\varepsilon}\iint f(v)\,f(v_1)(|v_1|^s+|v_1|^s)\,|v-v_1|^\beta\,dv\,dv_1\end{aligned} \tag{4.7}$$

where C_s and $K_{s,\varepsilon}$ are the constants from Theorem 4.1 multiplied by $\int_{S^2}h\,d\omega$ and $\int_{S^2}h_1\,d\omega$, respectively. For the negative term, first using (4.6) and the conservation of mass and then using Hölder's inequality gives

$$\begin{aligned}\iint &f(v)\,f(v_1)\,|v|^s\,|v-v_1|^\beta\,dv\,dv_1\\ &\geqslant c\int f(v)\,|v|^{s+\beta}\,dv \geqslant c\left(\int f(v)\,|v|^s\,dv\right)^{(s+\beta)/s}\end{aligned}$$

Because

$$|v_1|^{s-1}\,|v|\cdot|v-v_1|^{\beta}\leqslant|v_1|^{s-1}\,|v|(1+|v|+|v_1|)\leqslant(1+|v_1|^{s})(1+|v|^{2})$$

the positive term in (4.7) is bounded by

$$C_s(1+E_0)\left(\int f(v)\,|v|^s\,dv+1\right)$$

Writing $Y(t)=\int f(t,v)\,|v|^s\,dv$, we find

$$\frac{dY}{dt}\leqslant C_s(1+E_0)(Y+1)-2cK_{s,\varepsilon}\,Y^{1+\beta/s}$$

The result now follows by comparing this with the Bernoulli differential equation

$$\frac{d\tilde{Y}}{dt}=2A\,\tilde{Y}-B\,\tilde{Y}^{1+\beta/s}$$

which has the solution

$$\tilde{Y}(t)=\left[\tilde{Y}(0)^{-\beta/s}\,e^{-2A\beta t/s}+\frac{B}{2A}\,(1-e^{-2A\beta t/s})\right]^{-s/\beta}$$

If $Y(0)\leqslant 1$, there is really nothing to prove. Otherwise take $A=C_s(1+E_0)$ and $B=2cK_{s,\varepsilon}$.

Remark. The same general result holds for any kernel $B(\cos\theta, |v-v_1|)$ which is unbounded in $|v-v_1|$, as well as for noncutoff interactions, but the singularity of $C(t)$ as $t\to 0$ is stronger for a more slowly increasing B.

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